On the string solution in the SUSY - Skyrme model

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Abstract: In this paper, we have found the string solution in the SUSY Skyrme model. Moreover, the mechanics of decay of SUSY - string was discussed.

Keywords : String, SUSY, Skyrme model.

1 Introduction

String - like solution firstly obtained [2] from equaton of motion of the Skyrme model with a pion's mass term by A. Jackson. This string-like solution may be closely related to QCD string [2]. This string solution is unstable and it may decay by emitting pions [2]. During the decay a baryon current flows along the string, producing half a baryon and half an antibaryon. Long strings can decay via many different decay models, some producing baryon-antibaryon pairs [2]. Recently, M. Nitta and M. Skiiki constructed non-topological string solutions with $U(1)$ Noether charge in the Skyrme model with a pion mass term. And they also showed that this string were not stabilized by $U(1)$ rotation and decay into baryon-antibaryon pairs or mesons in the same way as the string without the charge. They showed that a rotating confugutation would reduce its rotational energy by emmiting pions. When this string becomes longer than $\pi/\hat{m}_\pi$, it will decay by emitting pions [3]. In this paper we want to connect idea of the string [2] to the supersymmetric skyrme model proposed by E. A. Bershoeff at el. [4]. This way may give us the SUSY string solution (superstring) and a mechanics of decay squark-antisquark, baryon-antibaryons, etc.

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2 The SUSY string solution

Let us consider the lagrangian with a mass of pion [8]

$$\mathcal{L} = -\frac{F_\pi^2}{16}Tr\left(\partial_\mu U^\dagger \partial^\mu U\right) + \frac{1}{32e^2}Tr\left(\left[U^\dagger \partial_\mu U, U^\dagger \partial_\nu U\right]^2\right) + \frac{1}{8}m_\pi^2 F_\pi^2 Tr\left[1-U\right]. \quad (1)$$

In terms of complex scalars A_i [4], this equation can be rewritten as

$$\mathcal{L} = \mathcal{L}_{susy} + \frac{1}{8}m_\pi^2 F_\pi^2\left[\bar{A}_1 + A_1 - 2\right]. \quad (2)$$

Now, let us consider the rotating soliton developed by M. Nitta at el. [3] from the original soliton constructed by A. Jackson [2]

$$U = \begin{bmatrix} \cos f\left(r\right) & i\sin f\left(r\right)e^{-i(\theta+\alpha(t))} \\ i\sin f\left(r\right)e^{i(\theta+\alpha(t))} & \cos f\left(r\right) \end{bmatrix} \quad (3)$$

$$\rightarrow A_1 = \cos f\left(r\right); A_2 = i\sin f\left(r\right)e^{i(\theta+\alpha(t))}, \quad (4)$$

in the cylindrical coordinate system with the metric

$$ds^2 = -dt^2 + dz^2 + dr^2 + r^2 d\theta^2. \quad (5)$$

Substituting this solution into Lagrangian (2), we obtain the string tension (see more in the appendix)

$$\mathcal{E} = \int 4\pi r^2 dr\left\{\tfrac{-f_\pi^2}{8}\left[\left(-\dot{\alpha}^2 + \tfrac{1}{r^2}\right)\sin^2 f + f'^2\right] + \right.$$

$$+\tfrac{1}{8e^2}\left[(a-b)\left(\left(\dot{\alpha}^4 - 2\tfrac{\dot{\alpha}^2}{r^2} + \tfrac{1}{r^4}\right)\sin^4 f + 2f'^2\sin^2 f\left(\dot{\alpha}^2 + \tfrac{1}{r^2}\right) + f'^4\right) + \right.$$

$$\left.\left. +b\left(\left(\dot{\alpha}^4 + \ddot{\alpha}^2 + 1\right)\sin^2 f + f''^2 + f'^4\right)\right] + \frac{1}{4}m_\pi^2 f_\pi^2\left(1-\cos f\right)\right\}. \quad (6)$$

Setting the dimensionless variable $\rho = f_\pi e r \equiv \gamma r$, and taking the variations of $f\left(\rho\right)$ in the string tension $\delta_f \mathcal{E} = 0$, we have the Euler-Lagrange equation

$$\tfrac{\partial}{\partial\rho}\tfrac{\delta\mathcal{E}}{\delta f'} - \tfrac{\delta\mathcal{E}}{\delta f} = 0$$

$$\rightarrow f''\left[-2\rho^2 + 4\rho^2(a-b)\sin^2 f\left(\tfrac{\dot{\alpha}^2}{\gamma^2} + \tfrac{1}{\rho^2}\right) + 12\rho^2 a f'^2\right] +$$

$$+f'^3\left[8\rho a - 2\sin 2f\left(\tfrac{\dot{\alpha}^2}{\gamma^2} + \tfrac{1}{\rho^2}\right)\right] + 4f'^2\rho^2(a-b)\sin 2f\left(\tfrac{\dot{\alpha}^2}{\gamma^2} + \tfrac{1}{\rho^2}\right) +$$

$$+f'\left[-4\rho + 8\rho\left(a-b\right)\sin^2 f\left(\tfrac{\dot{\alpha}^2}{\gamma^2} + \tfrac{1}{\rho^2}\right) - 2\sqrt{\rho}\left(a-b\right)\sin^2 f - \right.$$

$$-2\rho^2\left(a-b\right)\left(\tfrac{\dot{\alpha}^4}{\gamma^4}-\tfrac{2\dot{\alpha}^2}{\gamma^2\rho^2}+\tfrac{1}{\rho^4}\right)\sin 2f\sin^2 f\Big]+$$

$$+\rho^2\left[\frac{-b}{\gamma^4}\left(\ddot{\alpha}^2+\dot{\alpha}^4+1\right)+\frac{\dot{\alpha}^2}{\gamma^2}+\frac{1}{\rho^2}\right]\sin 2f-2\frac{m_\pi^2}{\gamma^2}\rho^2\sin f=0. \quad (7)$$

For the finiteness and regularity conditions of the string tension, one requires

$$f\left(0\right)=n\pi;\, f\left(\infty\right)=0, \quad (8)$$

where n is a positive integer. This field equation can be solved numerically with two boundary conditions (8).

2.1 the case of $b=0$

In the configuration of rotating soliton $\alpha\left(t\right)$ is angular rotation of soliton in time [2, 3] thus the quantity $\hat{\omega}$ defined as $\dot{\alpha}=\frac{\omega}{\gamma}\equiv\hat{\omega}$ will be understood as an angular velocity of rotating soliton. To obtain the asymptotic form of the profile $f\left(\rho\right)$ as $\rho\rightarrow\infty$ we need linearize the field equation (7). Setting $f=\delta f$, we get

$$\rho^2\delta f''+2\rho\delta f'+\left[1-\rho^2 m^2\right]\delta f-=0, \quad (9)$$

where $m^2=\left(\frac{m_\pi^2}{\gamma^2}-\hat{\omega}^2\right)$. This is the Bessel equation, to obtain solutions of Bessel function, we require

$$0<\hat{\omega}<\hat{m}_\pi, \quad (10)$$

where $\hat{m}_\pi=\frac{m_\pi}{\gamma}$.

Let us mention in ref. [3], the condition for angular velocity was obtained as $0<\hat{\omega}<\frac{\hat{m}_\pi}{\sqrt{2}}$ and they shown the mechanics of emitting pions of string when $\hat{\omega}$ increases over the critical value $\hat{\omega}_+=\hat{m}_\pi/\sqrt{2}$. However, in SUSY case, our critical value is $\hat{\omega}_+^{susy}=\hat{m}_\pi=\sqrt{2}\hat{\omega}_+$. In cases of critical values are larger than this critical value, the SUSY - String solution can be emitted into baryons - sbaryons pairs and pions - spions.

2.2 the case of $b\neq 0$

For this case, we have the equation

$$\delta f''+\frac{2}{\rho}\delta f'+\frac{1}{\rho^2}\delta f+\left[-\frac{b}{\gamma^2}\left(\frac{d\hat{\omega}}{dt}\right)^2-b\hat{\omega}^4+\hat{\omega}^2-\frac{b}{\gamma^4}-\hat{m}_\pi^2\right]\delta f=0. \quad (11)$$

We now consider an angular velocity of soliton is constant, means $\frac{d\omega}{dt}=0$. Similarly to a case of $b=0$, we require

$$\left[-b\hat{\omega}^4+\hat{\omega}^2-c\right]>0, \quad (12)$$

where $c = \frac{b}{\gamma^4} + \hat{m}_\pi^2$. Following conditions for b and ω are given as

$$-\gamma^2 m_\pi^2 < b < 0 \rightarrow 0 < \omega < \sqrt{x_2}, \tag{13}$$

$$0 < b < \frac{\gamma^2\left(-1+\sqrt{1+m_\pi^2}\right)}{2} \rightarrow \sqrt{x_1} < \omega < \sqrt{x_2}, \tag{14}$$

where $x_1 = \frac{-1-\sqrt{1-4bc}}{-2b}, x_2 = \frac{-1+\sqrt{1-4bc}}{-2b}$. Eqs (13), (14) show us areas of ω in which SUSY string may be decay. This is a different point between non-SUSY string [2, 3] and SUSY string.

3 Appendix

3.1 Recall the SUSY Skyrme model

Let us consider the SUSY Lagrangian Skyrme [4, 20]

$$\mathcal{L} = -\frac{f_\pi^2}{16} Tr\left(\partial_\mu U^\dagger \partial^\mu U\right) + \frac{1}{32e^2} Tr\left(\left[U^\dagger \partial_\mu U, U^\dagger \partial_\nu U\right]^2\right), \tag{15}$$

where U is can $SU(2)$ matrix, f_π is the pion decay constant, e is a free parameter. The ordinary derivatives in the Lagrangian density (15) is replaced by the covariant derivatives

$$\partial_\mu U \rightarrow D_\mu U = \partial_\mu U - iV_\mu U\tau_3, \tag{16}$$

Eq (15) becomes as

$$\mathcal{L} = -\frac{f_\pi^2}{16} Tr\left(D_\mu U^\dagger D^\mu U\right) + \frac{1}{32e^2} Tr\left(\left[U^\dagger D_\mu U, U^\dagger D_\nu U\right]^2\right). \tag{17}$$

Eq (17) is invariant under the local $U\left(1\right)_R$ and the global $SU\left(2\right)_L$ transformations

$$\begin{aligned} U\left(r\right) &\rightarrow AU\left(r\right)e^{i\lambda(r)\tau_3}, A \in SU\left(2\right)_L\,, \\ V_\mu\left(r\right) &\rightarrow V_\mu\left(r\right) + \partial_\mu\lambda\left(r\right). \end{aligned} \tag{18}$$

where the gauge field $V_\mu\left(r\right)$ is defined as

$$V_\mu = -\frac{i}{2} Tr\left(U^\dagger \partial_\mu U\tau_3\right). \tag{19}$$

One parametrizes the $SU(2)$ matrix U in terms of the complex scalars A_i

$$U\left(r\right) = \begin{pmatrix} A_1 & -A_2^* \\ A_2 & A_1^* \end{pmatrix}, \tag{20}$$

where $\bar{A}^i A_i = A_1^* A_1 + A_2^* A_2 = 1$. Eq (16) can be rewritten as

$$D_\mu A_i = (\partial_\mu - iV_\mu)\, A_i \ ,$$
$$D_\mu \bar{A}_i = (\partial_\mu + iV_\mu)\, \bar{A}_i \tag{21}$$

and the new form of gauge field is

$$V_\mu\,(r) = -\frac{i}{2}\bar{A}^i \overleftrightarrow{\partial} A_i. \tag{22}$$

Finally, we obtain Lagrangian in terms of complex scalars A_i

$$\mathcal{L} = -\frac{f_\pi^2}{8}\bar{D}_\mu \bar{A} D^\mu A - \frac{1}{16e^2}F_{\mu\nu}^2, \tag{23}$$

where $F_{\mu\nu}\,(V) = \partial_\mu V_\nu - \partial_\nu V_\mu$ One supersymmetrised Skyrme model by extending A_i to chiral scalar multiplets $(A_i, \psi_{\alpha i}, F_i)$ $(i, \alpha = 1, 2)$ and the vector $V_\mu\,(x)$ to real vector multiplets $(V_\mu, \lambda_\alpha, D)$. Here, the fields F_i are complex scalars, D is real scalar, $\psi_{\alpha i}$, λ_α are Majorana two-component spinors. $\psi_{\alpha i}$ corresponds to a left-handed chiral spinor, $\bar{\psi}^{\alpha i} = (\psi_i^\alpha)^*$ corresponds to a right-handed one. The SUSY Lagrangian density is given by

$$\mathcal{L}_{susy} = \tfrac{f_\pi^2}{8}\Big[-D^\mu \bar{A}^i D_\mu A_i - \tfrac{1}{2} i \bar{\psi}^{\dot{\alpha} i}\,(\sigma_\mu)_{\alpha\dot{\alpha}}\, \overleftrightarrow{D}^\mu \psi_i^\alpha + \bar{F}^i F_i -$$
$$-i\bar{A}^i \lambda^\alpha \psi_{\alpha i} + iA_i \bar{\lambda}^{\dot{\alpha}} \bar{\psi}^i_{\dot{\alpha}} + D\left(\bar{A}^i A_i - 1\right)\Big] +$$
$$+\frac{1}{8e^2}\left[-\frac{1}{2}F_{\mu\nu}^2 - i\bar{\lambda}^{\dot{\alpha}}\,(\sigma^\mu)^\alpha_{\dot{\alpha}}\,\partial_\mu \lambda_\alpha + D^2\right]. \tag{24}$$

This Lagrangian is invariant under the following set of supersymmetric transformations

$$\delta A_i = -\varepsilon^\alpha \psi_{\alpha i}, \tag{25}$$
$$\delta \psi_{\alpha i} = -i\bar{\varepsilon}^{\dot{\alpha}}\,(\sigma^\mu)_{\alpha\dot{\alpha}}\, D_\mu A_i + \varepsilon_\alpha F_i, \tag{26}$$
$$\delta F_i = -i\bar{\varepsilon}^{\dot{\alpha}}\,(\sigma^\mu)^\alpha_{\dot{\alpha}}\, D_\mu \psi_{\alpha i} - i\bar{\varepsilon}^{\dot{\alpha}} A_i \bar{\lambda}_{\dot{\alpha}}, \tag{27}$$
$$\delta V_\mu = -\frac{1}{2} i\,(\sigma_\mu)^{\alpha\dot{\alpha}}\left(\bar{\varepsilon}_{\dot{\alpha}}\lambda_\alpha + \varepsilon_\alpha \bar{\lambda}_{\dot{\alpha}}\right), \tag{28}$$
$$\delta\lambda_\alpha = \varepsilon^\beta\,(\sigma^{\mu\nu})_{\beta\alpha}\, F_{\mu\nu} + i\varepsilon_\alpha D, \tag{29}$$
$$\delta D = \frac{1}{2}\,(\sigma^\mu)_{\alpha\dot{\alpha}}\,\partial_\mu\left(\bar{\varepsilon}^{\dot{\alpha}}\lambda^\alpha - \varepsilon^\alpha \bar{\lambda}^{\dot{\alpha}}\right). \tag{30}$$

The field equation and their supersymmetric transformations lead to the following constraints

$$\bar{A}^i A_i = 0, \tag{31}$$
$$\bar{A}^i \psi_{\alpha i} = 0, \tag{32}$$

$$\bar{A}^i F_i = 0, \tag{33}$$

and following algebraic expressions for

$$V_\mu = -\frac{1}{2}\left\{i\bar{A}^i \overleftrightarrow{\partial}_\mu A_i + (\sigma_\mu)^{\alpha\dot{\alpha}}\,\bar{\psi}^i_{\dot{\alpha}}\psi_{\alpha i}\right\}, \tag{34}$$

$$\lambda_\alpha = -i\left\{\bar{F}^i\psi_{\alpha i} + i\,(\sigma^\mu)_{\alpha\dot{\alpha}}\,(D_\mu A_i)\,\bar{\psi}^{\dot{\alpha}i}\right\}, \tag{35}$$

$$D = D^\mu \bar{A}^i D_\mu A_i + \frac{1}{2} i\bar{\psi}^{\dot{\alpha}i}\,(\sigma^\mu)_{\alpha\dot{\alpha}}\left(\overleftrightarrow{D}_\mu \psi^\alpha_i\right) - \bar{F}^i F_i. \tag{36}$$

To obtain the minimum of SUSY extension, one set $\psi_{\alpha i} = F_i = 0$. Therefore, Eq (24) becomes as

$$\mathcal{L}_{susy} = -\frac{f_\pi^2}{8}\bar{D}^\mu \bar{A} D_\mu A + \frac{1}{8e^2}\left[-\frac{1}{2}F^2_{\mu\nu} + \left(\bar{D}^\mu \bar{A} D_\mu A\right)^2\right]. \tag{37}$$

However, there is another four-derivatives term of A_i, one gave the general form of SUSY Lagrangian

$$\mathcal{L}_{susy} = -\frac{f_\pi^2}{8}\bar{D}^\mu \bar{A} D_\mu A + \frac{1}{8e^2}\left\{a\left[-\frac{1}{2}F^2_{\mu\nu} + \left(\bar{D}^\mu \bar{A} D_\mu A\right)^2\right] + b\left\{\Box\bar{A}\Box A - \left(\bar{D}^\mu \bar{A} D_\mu A\right)^2\right\}\right\}, \tag{38}$$

where a, b are constants, $\Box = D^\mu D_\mu$ is the gauge covariant $D'alembertian$.

3.2 Some main results for Eq (6)

*** The first term**

We have

$$D_\mu A_i = (\partial_\mu - iV_\mu)\,A_i = \left[\partial_\mu + \frac{1}{2}\left(\partial_\mu \bar{A}^i\right)A_i + \frac{1}{2}\bar{A}^i\left(\partial_\mu A_i\right)\right]A_i, \tag{39}$$

$$\bar{D}^\mu \bar{A}^i = (\partial^\mu + iV^\mu)\,\bar{A}^i = \left[\partial^\mu - \frac{1}{2}\left(\partial^\mu \bar{A}^i\right)A_i - \frac{1}{2}\bar{A}^i\left(\partial^\mu A_i\right)\right]\bar{A}^i. \tag{40}$$

Inserting forms of A_i (4), let us final results

$$D_0 A_i = -\dot{\alpha}\sin f e^{i(\theta+\alpha)}\ ;\ D_1 A_i = 0\ ;\ D_2 A_i = i\cos f f' e^{i(\theta+\alpha)};\ D_3 A_i = -\sin f e^{i(\theta+\alpha)}\ ,$$

$$\bar{D}^0 \bar{A}_i = \dot{\alpha}\sin f e^{-i(\theta+\alpha)};\ \bar{D}^1 \bar{A}_i = 0;\ \bar{D}^2 \bar{A}_i = -i\cos f f' e^{-i(\theta+\alpha)};$$
$$\bar{D}^3 \bar{A}_i = -\sin f e^{-i(\theta+\alpha)}\ .$$

$$\Rightarrow \bar{D}^\mu \bar{A}_i D_\mu A_i = \left(-\dot{\alpha}^2 + 1\right)\sin^2 f + f'^2. \tag{41}$$

*** The second term**

We have

$$\left(F_{\mu\nu}\right)^2 = \left(\partial_1 V_2 - \partial_2 V_1\right)\left(\partial^1 V^2 - \partial^2 V^2\right), \tag{42}$$

in terms of forms of A_i (4), we obtain final results

$$F_{12}^2 = F_{21}^2 = 0;\ F_{13}^2 = F_{31}^2 = 0;\ F_{23}^2 = F_{32}^2 = 0\ .$$
$$\Rightarrow (F_{\mu\nu})^2 = 0 \tag{43}$$

*** The third term**
We have

$$\Box = D^\mu D_\mu = (\partial^\mu - iV^\mu)(\partial_\mu - iV_\mu)$$
$$= \partial^\mu\partial_\mu - i\partial^\mu V_\mu - iV^\mu\partial_\mu - V^\mu V_\mu = \partial^\mu\partial_\mu \tag{44}$$

Or

$$\Box\bar{A}\Box A = \partial_0^2\bar{A}^i\partial_0^2A_i + \partial_1^2\bar{A}^i\partial_1^2A_i + \partial_2^2\bar{A}^i\partial_2^2A_i + \partial_3^2\bar{A}^i\partial_3^2A_i. \tag{45}$$

Final results are

$$\partial_0^2\bar{A}^i\partial_0^2A_i = \cos^2 f\left(f'^4\right) + 2\sin 2f\left(f'^2\right)f'' + \sin^2 f\left(f''^2\right) + \sin^2 f\left(\ddot{\alpha}^2 + \dot{\alpha}^4\right), \tag{46}$$

$$\partial_1^2\bar{A}^i\partial_1^2A_i = 0, \tag{47}$$

$$\partial_2^2\bar{A}^i\partial_2^2A_i = \sin^2 f\left(f'^4\right) - 2\sin 2f\left(f'^2\right)f'' + \cos^2 f\left(f''^2\right), \tag{48}$$

$$\partial_3^2\bar{A}^i\partial_3^2A_i = \sin^2 f. \tag{49}$$

$$\Rightarrow \Box\bar{A}\Box A = \sin^2 f\left[\dot{\alpha}^4 + \ddot{\alpha}^2 + 1\right] + f''^2 + f'^4 \tag{50}$$

4 Conclusion

In this paper, we have performed analytic calculations, new results were found. In near future, they will be computed clearly by numerical methods, this way will give us the brilliant picture of mechanics of SUSY string's decay.

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